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Spatiotemporal pulse collapse on periodic potentials

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We present analytical results, based on the separation of different spatial scales and integral relations, which describe the spatiotemporal evolution of pulses in a nonlinear medium with periodically varying parameters (e.g., a nonlinear waveguide with a grating). Exact sufficient criteria for blow-up are formulated and the dependence of the spatiotemporal pulse collapse on the grating amplitude and period is discussed.

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As is well known, optical pulses may collapse simultaneously and symmetrically in time and space under the combined effect of diffraction, dispersion, and nonlinearity. Such *spatiotemporal self-focusing* of short but intense optical pulses has been recently discussed in Refs. [1–3]. The effect of group-velocity dispersion (GVD) has different influences on the pulse collapse depending on the sign of the GVD. In the case of normal GVD (see, e.g., Ref. [3]), a single pulse breaks up into two pulses that collapse with simultaneous subdivision into "cavities" of smaller scales. However, a pulse develops a singularity at a finite propagation distance only in the three-dimensional case. This type of collapse has been analyzed earlier in the context of wave propagation in plasmas [4], where the term "fractal collapse" has been introduced for the process whereby each small-scale bunch is, in turn, unstable to split into smaller-scale structures being compressed by collapse. In the case of anomalous GVD, spatiotemporal self-focusing of a nonlinear pulse takes place without breakup, and the pulse collapses as a whole even in the two-dimensional problem (see, e.g., [1]).

An interesting problem is the spatiotemporal nonlinear pulse dynamics in a medium with a periodically varying refractive index [5,6]. Nonlinear wave propagation in periodic structures has been a topic of extensive studies because it is a problem of fundamental interest and it has many potential practical applications. Interplay between the effects produced by dispersion and nonlinearity in the presence of periodic modulation of the refractive index results in a variety of interesting nonlinear phenomena, including so-called gap solitons (see, e.g., [7–9]). Spatiotemporal evolution of a pulse in a nonlinear waveguide with a periodic refractive-index profile has been recently studied using a variational approach and numerical simulations [5]. The variational approach is known to be an appropriate and simple way to describe general features of the nonlinear wave evolution. At the same time, this is not a rigorous method, because results are rather sensitive to the successful choice of trial functions. The optimum solution is to combine such a variational approach with numerical calculations and exact methods such as the virial theorem [10].

The purpose of this paper is to present exact sufficient criteria for spatiotemporal pulse collapse in a periodic medium for the limits of rapidly and slowly varying (one-dimensional) periodic modulation of the medium's parameters. As a matter of fact, these two limit cases may be considered as two intermediate steps of one process when a collapsing pulse changes an effective ratio of the pulse width to the period of the spatial modulation. In the first case of rapid oscillations, we apply the recently formulated asymptotic method [11] to separate fast and slow spatial variables, and show that, to the second-order terms, the resulting ("averaged") equation is a renormalized nonlinear Schrödinger (NLS) equation for which the collapse criterion may be found in an explicit form. When the pulse width becomes comparable to the periodic modulation, the effective averaging method cannot be applied to calculate the collapse criterion, and we use an approach based on a majoring function (see, e.g., Ref. [6]). A combination of both these approaches allows us to present sufficient criteria and to describe features of spatiotemporal collapse in a nonlinear medium with periodically varying parameters. A typical physical problem for which our results may be applied is the spatiotemporal pulse compression in a nonlinear Kerr-type waveguide with a periodic (one-dimensional) modulation of the refractive index (known as a grating).

As is well known, the spatiotemporal evolution of the dimensionless envelope $u(z,x,t)$ of the electromagnetic field in a waveguide with a periodic refractive index is governed by the following NLS equation:

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial t^2}\right)+|u|^2u=\epsilon\cos(kx)u, \qquad (1)$$

where we have used normalized variables (see, e.g., Ref. [5]) and selected the case of the anomalous GVD regime.

Equation (1) is a modification of the two-dimensional (2D) NLS equation. The main properties of the 2D NLS equation are rather well established (see, for example, Ref. [12] for a review). In particular, a solution of the 2D NLS equation may develop a singularity at a finite propagation distance [10,13]. This result was proved by means of the virial theorem [10]. The singularity occurs at a finite distance when the pulse power exceeds a certain critical value. More precisely, the necessary condition for the blow-up is that the "pulse energy" $\int |u(x,t)|^2\,dx\,dt$ of the input field distribution (at $z=0$) must be larger than that of the same integral calculated for the radial stationary solution of the lowest order. As was mentioned in [5,6], the similar features of the collapse dynamics may be also expected in the case of a periodic modulation of the refractive index. In this work, we analyze the effect of the periodic grating on the collapse dynamics for rapidly and slowly varying modulations (in comparison with the pulse scales) of the refractive index.

In the first case we assume that the parameter k in Eq. (1) is large and apply an averaging method recently used to analyze different regimes of the soliton dynamics in the presence of rapidly varying periodic perturbations [11,14,15]. A general approach for deriving an averaged equation to any order of asymptotic expansion has recently been proposed in Ref. [11]. Here we show that the method described in [11] may be also applied to the case of *periodic spatial perturbations*, provided the perturbation is rapidly varying, i.e., its spatial scale is much smaller than the characteristic width of the spatially localized solution of the problem under consideration.

Let us consider the periodic potential in Eq. (1) as rapidly oscillating, i.e., k is assumed large, and apply the method of the scale separation. In order to derive an effective equation for the slowly varying part of the wave field, we look for a general solution of Eq. (1) in the form of the following asymptotic expansion:

$$u = U + A\cos(kx) + B\sin(kx) + C\cos(2kx) + D\sin(2kx) + \cdots, \tag{2}$$

where the functions U and A, B, ... are assumed to be slowly varying, and U has the sense of the mean value of the field u on the spatial scale $\sim k^{-1}$. Our goal is to derive an effective equation for the function U. To this end, we substitute Eq. (2) into Eq. (1) and equate the coefficients in front of the different harmonics obtaining the infinite chain of coupled equations,

$$i\frac{\partial U}{\partial z} + \frac{1}{2}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial t^2}\right) + |U|^2U + \left(\frac{1}{2}U^*A^2 + U|A|^2 + \cdots\right) = \frac{1}{2}\epsilon A, \tag{3}$$

$$i\frac{\partial A}{\partial z} + \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \frac{1}{2}\left(\frac{\partial^2 A}{\partial x^2} - k^2A + 2k\frac{\partial B}{\partial x}\right) + \left(U^2A^* + 2|U|^2A + \cdots\right) = \epsilon\left(U + \frac{1}{2}C + \cdots\right), \tag{4}$$

$$i\frac{\partial B}{\partial z} + \frac{1}{2}\frac{\partial^2 B}{\partial t^2} + \frac{1}{2}\left(\frac{\partial^2 B}{\partial x^2} - k^2B - 2k\frac{\partial A}{\partial x}\right) + \left(U^2B^* + 2|U|^2B + \cdots\right) = \frac{1}{2}\epsilon D, \tag{5}$$

$$i\frac{\partial C}{\partial z} + \frac{1}{2}\frac{\partial^2 C}{\partial t^2} + \frac{1}{2}\left(\frac{\partial^2 C}{\partial x^2} - 4k^2C + 4k\frac{\partial D}{\partial x}\right) + \left(U^2C^* + \frac{1}{2}U^*A^2 + U|A|^2 + 2|U|^2C + \cdots\right) = \frac{1}{2}\epsilon A, \tag{6}$$

and so on. One of the main steps in our analysis is to find the appropriate form of asymptotic expansion for the coefficients A, B,..., which will allow us to solve the coupled equations (4)–(6). In this case, it may be proved (see, e.g., similar discussion in Ref. [11]) that the asymptotic expansions may be taken in the form

$$A = \frac{a_1}{k^2} + \frac{a_2}{k^4} + \cdots,\quad B = \frac{b_1}{k^3} + \cdots,$$
$$C = \frac{c_1}{k^4} + \cdots,\quad D = \frac{d_1}{k^5} + \cdots, \tag{7}$$

and so on. Substituting Eq. (7) into Eqs. (3)–(6) and equating terms of the same order in k^{-1}, we find

$$a_1 = -2\epsilon U, \tag{8}$$

$$b_1 = -2\frac{\partial a_1}{\partial x}, \tag{9}$$

$$a_2 = 2i\frac{\partial a_1}{\partial z} + \frac{\partial^2 a_1}{\partial t^2} - 3\frac{\partial^2 a_1}{\partial x^2} + 2U^2a_1^* + 4|U|^2a_1, \tag{10}$$

and so on. Applying now the expansions (7) to Eq. (3) we find that the equation for the slowly varying component U is written as

$$i\frac{\partial U}{\partial z} + \frac{1}{2}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial t^2}\right) + |U|^2U + \frac{1}{2}U^*\frac{a_1^2}{k^4} + U\frac{|a_1|^2}{k^4} = \frac{\epsilon}{2}\left(\frac{a_1}{k^2} + \frac{a_2}{k^4} + \ldots\right). \tag{11}$$

After using the results (8)–(10), this takes the form

$$i\frac{\partial U}{\partial z} + \frac{1}{2}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial t^2}\right) + |U|^2U = -\delta^2U - \frac{\delta^2}{k^2}\left(2i\frac{\partial U}{\partial z} + \frac{\partial^2 U}{\partial t^2} - 3\frac{\partial^2 U}{\partial x^2} + 12|U|^2U\right), \tag{12}$$

where we have introduced the dimensionless parameter $\delta^2 \equiv \epsilon^2/k^2$. It is important to note that the derivation presented above is still valid for large ϵ's provided k is large enough. As a matter of fact, it may be proven that Eq. (12) is valid for arbitrary δ up to the order of $O(1)$.

Thus, it follows from Eq. (12) that the averaged dynamics on the rapidly varying periodic potential is described by *a renormalized* NLS equation.

As is well known (see, e.g., the review paper [12]), the blow-up condition for the 2D NLS equation may be written as $H < 0$, where H is the Hamiltonian corresponding to the NLS equation. For Eq. (12) this yields $H_{ren} < 0$, where H_{ren} is the Hamiltonian of the renormalized NLS equation. This leads to the integral relation:

$$\int\int_{-\infty}^{\infty} dxdt \left(\left|\frac{\partial U}{\partial x}\right|^2 + \alpha \left|\frac{\partial U}{\partial t}\right|^2 - \beta |U|^4 \right) < 0, \quad (13)$$

where $\alpha = (k^2+2\delta^2)/(k^2-6\delta^2)$ and $\beta = (k^2+12\delta^2)/(k^2-6\delta^2)$ or for $\delta^2 \ll k^2$

$$\alpha \approx 1 + 8\frac{\delta^2}{k^2}, \quad \beta \approx 1 + 18\frac{\delta^2}{k^2}. \quad (14)$$

Therefore the renormalization of the NLS equation leads to a change of the collapse criterion compared to the primary (unrenormalized) model. This may allow us to avoid blow-up instabilities for input pulse with parameters lying in the region of the standard blow-up of the NLS model, without taking into account other physical mechanisms (such as saturation of the refractive index) that prevent blow-up.

If the blow-up condition is satisfied, the pulse is compressed due to collapse. Being compressed on the periodic potential, the pulse becomes so narrow that its width reaches a value comparable to the period of the potential. In this case, the assumption we made above is no longer valid, and, to predict the subsequent evolution of the pulse, we should develop another approach. The other approach we use here does not assume a rapidly varying potential, and the main contribution to the collapse criterion appears in the first-order approximation in the potential amplitude ϵ.

To start the analysis, we note that, in contrast to the 2D NLS equation, Eq. (1) does not conserve the x projection of the total momentum of the system, and this makes it impossible to apply the well-known procedure that has been used for the proof of collapse in the 2D NLS equation [10]. That is why we use here the so-called majoring equation method (see, e.g., [16]). In a simple way, this method may be formulated as follows. For the partial differential equation under consideration, one introduces an appropriate integral characteristic of its localized solutions. With the successful choice of such a quantity, one may derive an ordinary differential equation (or, at least, inequality) for this function. Solving this equation (or inequality), one may find sufficient conditions for a singularity to appear. In terms of this approach, the virial theorem [10] is, as a matter of fact, a particular case of a parabolic majoring function.

To prove the existence of collapse in Eq. (1), we consider as a majoring function the positively defined quantity $S = \int (x^2 + t^2)|u|^2 dxdt$, which has the sense of the effective (spatiotemporal) pulse width. It is easy to show that the evolution of this functional in z is described by the equation

$$\frac{d^2S}{dz^2} = 8H + 4\int_{-\infty}^{\infty} |u|^2 \left[2f(x) + x\frac{df(x)}{dx}\right] dx\, dt, \quad (15)$$

where $f(x)$ stands for the spatial function in the right-hand side of Eq. (1), and H is the system Hamiltonian which is a conserved quantity. The result (15) is rather general; we apply it to the particular case $f(x) = \epsilon \cos(kx)$ to make a natural link to the case already analyzed above when pulses, being compressed due to blow-up, have widths of order of the period of the potential.

The right-hand side of Eq. (15) may be estimated by means of the Hölder inequality (see, e.g., Refs. [6,16]). As a result, we may obtain the following differential inequality on S:

$$\frac{d^2S}{dz^2} \leq 8(H + \epsilon N) + 4\epsilon k S^{1/2} N^{1/2} \equiv -\frac{\partial W(S)}{\partial S}, \quad (16)$$

where $W(S) = AS - BS^{3/2}$, $A = -8(H + \epsilon N)$, $B = (8/3)\epsilon k N^{1/2}$, and $N = \int dxdt|u|^2$ is the other conserved quantity of Eq. (1) which has the sense of the system energy for optical problems. The representation (16) allows us to use the analogy with the motion of a particle in the effective potential $W(S)$ treating z as the "time" variable. This analogy adequately describes all aspects of the evolution of S.

First, let us consider the most important case of the "natural" initial conditions when there is no artificial initial focusing of the wave packet, namely, $(dS/dz)|_{z=0} = 0$. It is easy to show that under the condition

$$H + \epsilon N \leq 0, \quad (17)$$

the potential $W(S)$ has a single maximum at the point $S_m = (2A/3B)^2$. Let us choose, for simplicity but without loss of generality, the initial pulse satisfying the condition $S_0 \equiv S(0) < S_m$. Such a choice provides the condition $dW(S)/dS > 0$ to be satisfied, so that dS/dz is a monotonically decreasing function. From Eq. (16) it follows that

$$\frac{1}{2}\left(\frac{dS}{dz}\right)^2 \geq \frac{1}{2}\left(\frac{dS_0}{dz}\right)^2 + W(S_0) - W(S) \equiv E - W(S). \quad (18)$$

The above choice of the initial data provides also the condition $E > W_m \equiv W(S_m)$ to be satisfied. Resolving the differential inequality (18), we finally obtain

$$\int_S^{S_0} \frac{ds}{\sqrt{E - W(s)}} \geq \sqrt{2}z. \quad (19)$$

It is clear that the "moment of time" when the particle reaches the point $S = 0$ corresponds to the collapse point. In other words, if the integral in the left-hand side of Eq. (19) converges, then the corresponding solution of Eq. (1) develops a singularity at a finite point z_0. Therefore, at $z = 0$ an input pulse satisfies the two following conditions:

$$H + \epsilon N \leq 0, \quad S(0) < S_m = \left(\frac{2A}{3B}\right)^2, \quad (20)$$

then the pulse blows-up at the point

$$z_0 \leq \int_0^{S_0} \left(ds/\sqrt{2[E-W(s)]}\right).$$

Thus we have proven that the positive function $S(z)$ becomes zero at the point z_0. Using this contradiction and the additional inequality $N^2 \leq SI$, where the quantity I is defined as $I = \int (|u_x|^2 + |u_t|^2) dx dt$, one may check that the integral characteristic I becomes unbounded at the point z_0 as well. The conditions (20) give us the sufficient integral criteria of collapse in Eq. (1), and this result is the analog of the virial theorem [10] that has been proven for the 2D NLS equation in the absence of the external potential.

Our results may be easily generalized to cover the case of *an arbitrary periodic potential* satisfying the following conditions: $\max|f(x)| \leq C_1$ and $\max|f_x(x)| \leq C_2$, $C_{1,2}$ being constant. In the general case, the majoring inequality for the quantity $S(z)$ is written as [cf. Eq. (16)]

$$\frac{d^2S}{dz^2} = 8H + 4\int_{-\infty}^{\infty} |u|^2 \left[2f(x) + x\frac{df(x)}{dx}\right] dx\, dt \leq 8(H + NC_1) + 4S^{1/2}N^{1/2}C_2, \quad (21)$$

and all the results follow directly from Eqs. (17)–(20) by the simple change $\epsilon \to C_1$ and $\epsilon k \to C_2$. Thus this approach is rather general, and it also may be applied for a fairly large class of external periodic potentials.

At last, we would like to mention that some of the results obtained in the present paper may be useful to understand, at least qualitatively, the mechanism of the collapse-induced energy localization in a discrete lattice (see [17–19]). Indeed, in the quasicontinuum approximation when the pulse width is much larger than the lattice spacing, the discreteness may be treated as an effective periodic potential to the pulse (see, e.g., [20]). As follows from the problem considered here, such a potential changes the condition for blow-up. In fact, such "a renormalized blow-up dynamics" has been recently described numerically by Bang *et al.* [17] for the discrete NLS equation. We would like to note that the conclusions of the present study are in a good qualitative agreement with those numerical results. In particular, it was shown in [17] that discreteness makes possible the blow-up even for subcritical power nonlinearities, and this is similar to the renormalization we have found for rapidly varying potentials. On the other hand, when the pulse is narrow enough the blow-up in lattices is suppressed by discreteness, and this is similar to the change of the collapse criterion we calculated with the help of the majoring function.

In conclusion, we have presented sufficient criteria for blow-up in the problem of spatiotemporal collapse on periodic potentials. As we have shown, the problem may be effectively treated in the limits of rapidly and slowly varying periodic potentials. In the former case, we have applied asymptotic expansions to calculate a renormalization of the system Hamiltonian and modification of the blow-up condition, and in the latter case, we have proven an analog of the virial theorem. We have shown that a periodic potential not only changes the sufficient criterion for blow-up but also its effect is very essential in the vicinity of the focusing point.

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